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APPROXIMATE SOLUTION OF THE LINEAR HEAT EQUATION IN HETEROGENEOUS MEDIA

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The asymptotic Vishik–Lyusternik method is used to solve the linear heat equations in heterogeneous media.

The heat equations in heterogeneous two-component continuous medium have the form [1]:

$$\varepsilon_1 \frac{\partial T_1}{\partial t} + \frac{T_1 - T_2}{\tau} = \varkappa \nabla^2 T_1, \quad \frac{\partial T_2}{\partial t} + \frac{T_2 - T_1}{\tau} = \varepsilon_2 \varkappa \nabla^2 T_2, \qquad (1)$$

$$\varepsilon_1 = m_1\gamma_1 c_1/m_2\gamma_2 c_2, \quad \varepsilon_2 = m_2\lambda_2/m_1\lambda_1.$$

Equations that are analogous to system (1) describe also the nonstationary filtration of a homogeneous liquid in crack-porous media [2, 3]. We always have the condition $\varepsilon_2 \ll 1$ [1, 4]; such a condition is not necessary for ε_1.

In the solution of system (1), one can put $\varepsilon_2 = 0$. However, the solution of the degenerate system thus obtained can differ considerably from the exact solution of the system (1) near the boundary of the region under study. This disagreement can be avoided by using the approximate Vishik–Lyusternik method for the solution of (1) [5, 6]. We consider the method on the example of a one-dimensional problem. In this case we rewrite (1) as

$$\varepsilon_1 \frac{\partial T_1}{\partial \theta} + T_1 - T_2 = \frac{\partial^2 T_1}{\partial y^2}, \quad \frac{\partial T_2}{\partial \theta} + T_2 - T_1 = \varepsilon_2 \frac{\partial^2 T_2}{\partial y^2}, \qquad (2)$$

$$\theta = t/\tau, \quad y = x/\sqrt{\varkappa\tau}.$$

The initial and boundary conditions are formulated as follows:

$$\begin{aligned} \theta = 0:&\quad T_1 = T_2 = 0, \\ y = 0:&\quad T_1 = T_2 = T_0 = \text{const}, \\ y \to \infty:&\quad T_1 = T_2 = 0. \end{aligned} \qquad (3)$$

According to the Vishik–Lyusternik method, the required solution is represented in the form of two terms

$$\tilde{T}_i = w_i + v_i, \quad i = 1, 2. \qquad (4)$$

Here w_i is the solution of the degenerate system (2):

$$\varepsilon_1 \frac{\partial w_1}{\partial \theta} + w_1 - w_2 = \frac{\partial^2 w_1}{\partial y^2}, \quad \frac{\partial w_2}{\partial \theta} + w_2 - w_1 = 0 \qquad (5)$$

with the following initial and boundary conditions:

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$$\begin{aligned} \theta = 0: &\quad w_1 = w_2 = 0, \\ y = 0: &\quad w_1 = T_0, \\ y \to \infty: &\quad w_1 = 0. \end{aligned} \tag{6}$$

It is seen from (5) and (6) that w_i does not satisfy all boundary conditions (3). Using v_i, it is necessary to transform the obtained solution in such a way that it satisfies all conditions (3). The term v_i behaves similarly to a boundary layer, i.e., it is appreciable near the boundary and decays fast along the normal directed toward the interior of the region.

By the elongation transformation

$$\xi = y/\sqrt{\varepsilon_2} \tag{7}$$

we introduce a local coordinate adjacent to the boundary y = 0. By omiting in (2) the terms which contain ε_2 as a coefficient, we obtain the following system for the determination of v_i:

$$\frac{\partial^2 v_1}{\partial \xi^2} = 0, \quad \frac{\partial v_2}{\partial \theta} + v_2 - v_1 = \frac{\partial^2 v_2}{\partial \xi^2}. \tag{8}$$

According to the above discussion, the initial and boundary conditions for v_i must be such that all conditions (3) are satisfied:

$$\begin{aligned} \theta = 0: &\quad v_2 = 0, \\ \xi = 0: &\quad v_1 = 0, \ v_2 = T_0 - w_2(0, \theta), \\ \xi \to \infty: &\quad v_1 = v_2 = 0. \end{aligned} \tag{9}$$

The Laplace–Carson transform of the solution of (5) and (6) has the form

$$L\{w_1\} = T_0 \exp(-\beta y), \quad L\{w_2\} = \frac{T_0}{1+\sigma} \exp(-\beta y), \tag{10}$$

$$\beta = \left[\frac{\sigma}{1+\sigma} + \varepsilon_1 \sigma\right]^{1/2}.$$

It is seen from (10) that

$$w_2(0, \theta) = T_0 [1 - \exp(-\theta)], \tag{11}$$

i.e., the obtained solution indeed does not satisfy all the conditions (3). It is immediately seen from (8) and (9) that $v_1 \equiv 0$. To determine v_2, we must solve the following equation:

$$\frac{\partial v_2}{\partial \theta} + v_2 = \frac{\partial^2 v_2}{\partial \xi^2} \tag{12}$$

with the conditions

$$\theta = 0, \quad \xi \to \infty: \ v_2 = 0, \quad \xi = 0: \ v_2 = T_0 \exp(-\theta). \tag{13}$$

The Laplace–Carson transform of the solution of (12) and (13) has the form

$$L\{v_2\} = T_0 \frac{\sigma}{1+\sigma} \exp(-\sqrt{1+\sigma}\, \xi), \tag{14}$$

where v_2 is the zero-order boundary layer function [5]. It is not difficult to write down the Laplace–Carson transform of the exact solution of (2) and (3):

$$L\{T_1\} = \frac{T_0}{2}\left[\left(1-\frac{a}{d}\right)\exp(-sy) + \left(1+\frac{a}{d}\right)\exp(-zy)\right],$$

$$L\{T_2\} = \frac{T_0}{2}\left[\left(1+\frac{b}{d}\right)\exp(-sy) + \left(1-\frac{b}{d}\right)\exp(-zy)\right]; \quad (15)$$

$$s = \left\{\frac{1}{2}\left[1+\varepsilon_1\sigma + \frac{1+\sigma}{\varepsilon_2} + \sqrt{\left(1+\varepsilon_1\sigma - \frac{1+\sigma}{\varepsilon_2}\right)^2 + \frac{4}{\varepsilon_2}}\right]\right\}^{1/2},$$

$$z = \left\{\frac{1}{2}\left[1+\varepsilon_1\sigma + \frac{1+\sigma}{\varepsilon_2} - \sqrt{\left(1+\varepsilon_1\sigma - \frac{1+\sigma}{\varepsilon_2}\right)^2 + \frac{4}{\varepsilon_2}}\right]\right\}^{1/2}, \quad (16)$$

$$a = 1+\varepsilon_2+(1-\varepsilon_1\varepsilon_2)\sigma, \quad b = 1+\varepsilon_2-(1-\varepsilon_1\varepsilon_2)\sigma,$$

$$d = \{[\varepsilon_2 - 1 - (1-\varepsilon_1\varepsilon_2)\sigma]^2 + 4\varepsilon_2\}^{1/2}.$$

We compare the obtained results. Since the "discrepancy" in the boundary layer decays with time (see (11)) it is sufficient to make the comparison for small times. Using the condition $\varepsilon_2 \ll 1$, it is not difficult to obtain from (15) and (4) $T_1 - \tilde{T}_1 \approx 0$, $T_2 - \tilde{T}_2 \approx 0$, i.e., the solution obtained by the Vishik–Lyusternik method agrees well with the exact solution. The solution (10) of the degenerate system for T_1 will also agree well with the exact solution (as $w_1 \equiv \tilde{T}_1$), and for T_2 we obtain

$$L\{T_2 - w_2\} = T_0 \exp(-y\sqrt{\sigma/\varepsilon_2}),$$

which corresponds to the original function [7]

$$T_2 - w_2 = T_0 \operatorname{erfc}\left(\frac{y}{2\sqrt{\varepsilon_2\theta}}\right). \quad (17)$$

It is easily seen from (17) that the difference $(T_2 - w_2)$ near the boundary $y = 0$ will not, in general, be a small quantity.

If we consider a finite segement (e.g., y = 0–1), the right-hand side of (4) should be supplemented by another term which is defined analogously to v_i, i.e., in the neighborhood of the boundary $y = 1$ we introduce the local coordinate $\eta = (1-y)/\sqrt{\varepsilon_2}$, and the initial and boundary conditions are formulated in an analogous fashion.

NOTATION

m_i, part of the volume occupied by the i-th component; γ_i, density, kg/m^3; c_i, specific heat per unit mass, J/kg·deg; λ_i, thermal conductivity coefficient, W/m·deg; $\varkappa$, thermal conductivity coefficient, m^2/sec; x, coordinate, m; t, time, sec; T_i, temperature, °K; τ, retardation time, sec; $L\{\ \}$, Laplace–Carson transform; and σ, parameter of the Laplace–Carson transform. The indices 1 and 2 refer to components with high and low thermal conductivity.

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